SYNTHETIC ESTIMATORS IN SMALL-SAMPLES

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A novel recursive bias method is proposed for dynamic panel data models to reduce the estimator bias without large N or T, or both. Recursively, it decomposes the estimator bias into systematic and random components. The application compares three different sets of estimators with different methodologies, all using a small sample size of 60 and Maximum Likelihood Estimation. The first set uses the novel recursive bias method with synthetic data, the second set uses the expo-power utility method using real data, and the third set uses the traditional asymptotic bias method based on Monte Carlo simulations. This comparison demonstrates that synthetic estimators gain efficiency and are closest to the "true parameter value". The novel method is less expensive in computational power and processing time than Monte Carlo simulations. As a result, the proposed method could be a feasible option to provide efficient estimators for robust statistical inference and decision making.

KEYWORDS: bias components; dynamic panel models; robust inference; small-samples; synthetic estimators.

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1. INTRODUCTION

In the literature, the traditional asymptotic bias method uses Monte Carlo simulations or bootstrap experiments to derive sensitivities related to panel data estimators and their asymptotic bias properties by enlarging N or T, or both. This method tries to reduce the estimator asymptotic bias to provide efficient estimators which are important for robust statistical inference. Subsequently, these papers state a procedure to reduce asymptotic bias, *i.e.*, Hsiao and Zhang (2015), Abadie and Imbens (2011), Hsiao and Tahmiscioglu (2008), Hsiao et al. (2002), and MacKinnon and Smith (1998).

In this literature, there are different asymptotic estimator bias properties depending on initial assumptions, functional forms, sample size, endogeneity treatments, and Maximum Likelihood Estimation (MLE), or Generalized Method of Moments (GMM) estimations. For example, according to Hsiao (2003), if the outcome variable is fixed and the intercept estimator measures individual specific effects, the MLE estimator is a covariance estimator. This work finds that this covariance estimator is asymptotically normally distributed with a mean of zero if N is fixed and T is large. Another example is provided by Hahn and Kuersteiner (2002), who show that the covariance estimator is asymptotically biased of order of the square root of N over T when both N and T approach infinity, provided that the ratio of T over N approaches a constant different from zero. Arellano and Bond (1991) find an efficient GMM estimator that is asymptotically unbiased if T is fixed, and N goes to infinity. Alvarez and Arellano (2003) report an asymptotically biased estimator of order of the square root of c^*, where c^* lies between zero and infinity and converges to itself when N tends to infinity.

Panel data involve two dimensions. The first is N, which represents the number of individuals, and the second is T, which represents the number of time periods. This paper proposes a novel recursive bias method, which does not require enlarging panel data dimensions N or T, or both, to provide efficient estimators. That is to say, N or T, or both remains fixed. This method applies to small-samples and treats bias as a type of serial correlation problem. Recursively, it decomposes the estimator bias serial correlation problem into systematic and random components, reducing in this way its bias toward zero.

Three competing methodologies for producing efficient estimators are reported and compared in Table I. They applied MLE on a small-sample size of 60 observations. The first column of Table I reports synthetic estimators computed with the novel recursive bias method. The second column reports real data estimators using the expo-power utility method. The third column reports Monte Carlo estimators computed with the traditional asymptotic bias method. This comparison shows that the recursive bias method estimators gain efficiency by displaying the smallest errors. The novel method requires less computational power and time compared to the asymptotic bias method. The recursive bias method runs on personal computers in five seconds without increasing the panel dimensions N or T, or both, while the asymptotic bias method needs multiple processors in specialized setups, taking days, weeks, or even months to execute. The asymptotic bias method is computationally expensive in enlarging and repeating N or T, or both, and in inverting the projection matrix multiple times. The recursive bias method is a feasible option for replacing the traditional asymptotic bias method for robust statistical inference and decision making.

This paper is organized as follows, section 2 introduces two dynamic panel data model assumptions. Section 3 provides a traditional asymptotic bias method representation. Here, T is large, since it goes to infinity. Section 4 presents the novel recursive bias method. Here, N or T remains fixed. Section 5 presents an application and comparison of different estimators and methods in Table I. In section 6 the conclusions are put forward.

2. DYNAMIC PANEL DATA MODEL ASSUMPTIONS

Suppose that a dynamic panel data econometric model has the following form:

$$y_{it} = \alpha_i + \beta y_{i,t-1} + u_{it}, \qquad i = 1, \ldots, N, \quad t = 1, \ldots, T. \tag{1}$$

where y_{it} is the dependent variable, α_i stands for individual fixed effects, β is an efficient first differencing estimator or the "true parameter value", $y_{i,t-1}$ is the dependent variable with one time lag, u_{it} is the error term, i represents the individual dimension, and t represents the time dimension. Here, the error term on equation (1) is equal to zero, as the error term has not yet being computed from the estimation of this econometric model. It is a convention to annotate the error term on the econometric model.

Once the dynamic panel data econometric model expressed on equation (1) is estimated, it yields,

$$y_{it} = \hat{\alpha_i} + \hat{\beta} y_{i,t-1} + \hat{u}_{it}, \qquad i = 1, \ldots, N, \quad t = 1, \ldots, T. \tag{2}$$

where the hat over the estimators expresses that the estimation has being executed. For example, $\hat{u}_{it}$ is the estimator for the error term expressed on the dynamic panel data econometric model. Once the estimation is executed $\hat{u}_{it}$ is a residual vector computed from the difference between y_{it} and $\hat{y}_{it}$.[1]

The estimator $\hat{\beta}$ has a serial correlation problem. This problem is due to individual specific effects present in $\hat{\alpha}_i$ and $\hat{\beta}$ estimators. Since $\hat{\beta}$ considers individual and time effects and $y_{i,t-1}$ has the two panel data dimensions *i* and $t-1$. In equation (2) $\hat{\beta}$ bias is a result of double counting individual-specific effects.[2]

Assumption 1. u_{it} in equation (1) is a random variable with distribution $N(0, I\sigma_u^2)$, where I is the identity matrix. For the rest of estimators presented in this note, their first and second moments fulfill normality conditions and their third and fourth moments are finite.[3]

A modified omitted variable formula is used to represent the expected value of $\hat{\beta}$ and its bias, as follows:[4]

$$E[\hat{\beta}|\hat{\alpha}_i] = \beta + \frac{cov[\hat{\alpha}_i, y_{i,t-1}]}{var[\hat{\alpha}_i]}\hat{u}_{it} \tag{3a}$$

After estimating equation (1), $E[\hat{\beta}|\hat{\alpha}_i]$ expresses an expected conditional mean of $\hat{\beta}_i$ given $\hat{\alpha}_i$, and $\frac{cov[\hat{\alpha}_i, y_{i,t-1}]}{var[\hat{\alpha}_i]}\hat{u}_{it}$ is assumed to represent this estimator bias.[5] Clearly, equation (3a) is nonlinear in its bias component.

$$E[\hat{\beta}|\hat{\alpha}_i] = \beta + \xi_{it} \tag{3b}$$

[1] The var-cov matrix for the residuals is computed as $\hat{u}_{it}$ times $\hat{u}_{it}$ transpose.

[2] This type of double counting is considered in MacKinnon et al. (2023). General panel VAR models are analyzed by Holtz-Eakin (1998).

[3] According to Douc et al. (2014) and Spanos (1999), fourth moment finiteness is associated with a stationary solution in a strict-sense. In Arellano and Bond (1991), the fourth-order indicates a lower or faster convergence to normality.

[4] As far as the author concerns, the celebrated omitted variable formula has not being criticized by presenting the estimator as a function of observations.

[5] For Makowski et al. (2006) the model $y = \alpha_1 x_1 + \alpha_2 x_2 + \varepsilon$ has the following omitted variable formula: $\hat{\alpha}_1 = \frac{\sum_{i=1}^N x_{1i} y_i}{\sum_{i=1}^N x_{1i}^2}$, and $E(\hat{\alpha}_1) = \alpha_1 + \frac{\sum_{i=1}^N x_{1i} y_i}{\sum_{i=1}^N x_{1i}^2}\alpha_2$. He uses notation for deviations in small samples, and a var-cov estimator representation.

where $\xi_{it} = \dfrac{cov[\hat{\alpha}_i, y_{i,t-1}]}{var[\hat{\alpha}_i]}\hat{u}_{it}$ is the estimator bias. This estimator bias will be treated through a modified Beveridge and Nelson (1981) decomposition. This decomposition linearizes the bias into two components, one systematic and the other random.

Assumption 2. ξ_{it} in equation (3b) is a bias serial correlation problem with two components. This bias serial correlation problem components are shown next:

$$\xi_{it} = \delta_i + \omega_{it} \tag{4}$$

where δ_i is individual fixed effects or the bias systematic component represented by its mean, ω_{it} is the bias random component having a serial correlation problem. Once a modified Beveridge and Nelson (1981) decomposition is applied equation (4) has two components.

3. THE TRADITIONAL ASYMPTOTIC BIAS METHOD

Assumption 2 applied to equation (4) identifies the systematic component as δ_i and the random component as ω_{it}. Here, $\psi(L)$ is a moving average estimator sequence of ξ_{it}, *i.e.*, $\psi(1), \cdots, \psi(T)$. Thus,

$$\omega_{it} = \psi(L)\xi_{it} \tag{5}$$

Here, ω_{it} is expressed as a moving average polynomial of order T. Expanding the moving average polynomial $\psi(L)$ leads to:[6]

$$\omega_{it} = \psi(1)\xi_{i,t-1} + \psi(2)\xi_{i,t-2} + \cdots + \psi(T)\xi_{i,0} \tag{6}$$

where $\psi(1)$ represents a moving average estimator of order one, $\psi(2)$ represents a moving average estimator of order two, and so on. Finally, $\psi(T)$ represents a moving average estimator of order T. Plugging equation (6) into equation (5) and (4) yields:

$$\xi_{it} = \delta_i + \psi(1)\xi_{i,t-1} + \psi(2)\xi_{i,t-2} + \cdots + \psi(T)\xi_{i,0} \tag{7}$$

Then, equation (3a) can be rewritten as

$$E[\hat{\beta}|\hat{\alpha}_i] = \beta + \delta_i + \psi(1)\xi_{i,t-1} + \psi(2)\xi_{i,t-2} + \cdots + \psi(T)\xi_{i,0} \tag{8}$$

[6] The introduction of this notation helps in maintaining parsimony within this paper sections.

The bias of $\hat{\beta}$ is represented by $\delta_i + \psi(1)\xi_{i,t-1} + \psi(2)\xi_{i,t-2} + \cdots + \psi(T)\xi_{i,0}$. Hayashi (2000) states that the Lindeberg-Levy Central Limit and the Multivariate Convergence in Distribution Theorems find an estimator as sequences of random variables that converges in distribution to $x \sim N(0, \Sigma)$, and $\sqrt{n}(\bar{z_n} - \mu) \underset{d}{\rightarrow} x$ only if x is efficient with an asymptotic bias equal to zero. This section explains the traditional asymptotic bias method, where T is large, since it goes to infinity.

$$\lim_{T\rightarrow\infty} (\delta_i + \psi(1)\xi_{i,t-1} + \psi(2)\xi_{i,t-2} + \cdots + \psi(T)\xi_{i,0}) = 0 \tag{9}$$

If the above theorems hold then equation (8) reduces to $E[\hat{\beta}|\hat{\alpha}_i] = \beta$, where β is an efficient estimator.

4. A NOVEL RECURSIVE BIAS METHOD

A novel method is proposed to find an efficient dynamic panel data estimator under assumptions 1 and 2. Next, theorems 1 and 2 and their proofs describe the recursive bias method.

THEOREM 1: A consistent and efficient synthetic estimator in the presence of specific individual fixed effects correlation is obtained by estimating its bias components.

PROOF: Plugging equation (8) into equation (1) provides:

$$y_{it} = \alpha_i + [\beta + \delta_i + \psi(1)\xi_{i,t-1} + \psi(2)\xi_{i,t-2} + \cdots + \psi(T)\xi_{i,0}]\, y_{i,t-1} + u_{it} \tag{10}$$

Distributing the $y_{i,t-1}$ term gives:

$$y_{it} = \alpha_i + \beta y_{i,t-1} + \delta_i y_{i,t-1} + \psi(1)\xi_{i,t-1} y_{i,t-1} + \psi(2)\xi_{i,t-2} y_{i,t-1} + \cdots + \psi(T)\xi_{i,0} y_{i,t-1} + u_{it} \tag{11}$$

Collecting the individual-effects estimators in only one term, $\eta_i = \alpha_i + \delta_i y_{i,t-1}$ yields:[7]

$$y_{it} = \eta_i + \beta y_{i,t-1} + \psi(1)\xi_{i,t-1} y_{i,t-1} + \psi(2)\xi_{i,t-2} y_{i,t-1} + \cdots + \psi(T)\xi_{i,0} y_{i,t-1} + u_{it} \tag{12}$$

[7] Here, δ_i is an individual fixed effects estimator. Although $y_{i,t-1}$ contains both data panel dimensions, δ_i considers only specific individual fixed effects. In fact, δ_i possesses two panel dimensions. This characteristic posses a challenge to obtain an efficient estimator.

Equation (12) represents the first iteration of the proposed method to separate and quantify the bias components. Consider the term $\psi(1)\xi_{i,t-1}y_{i,t-1}$. Its estimator can be decomposed into systemic and random components using a modified Beveridge-Nelson decomposition, as in equation (4).

$$E[\hat{\psi}(1)|\hat{\eta}_i] = \psi(1) + \frac{cov[\hat{\eta}_i, \xi_{i,t-1}y_{i,t-1}]}{var[\eta_i]}\hat{u}_{it} \tag{13}$$

Assumption 2 applied to equation (13) shows that the systematic component is $\psi(1)$ and the random component is $\frac{cov[\hat{\eta}_i, \xi_{i,t-1}y_{i,t-1}]}{var[\eta_i]}\hat{u}_{it}$. Next, the analogs of equations (4)-(11) are presented for the $\psi(1)$ estimator. For simplicity and comparison, let the $\psi(1)$ bias be represented as follows. The underline represents the first iteration.

$$\underline{\xi}_{it} = \frac{cov[\hat{\eta}_i, \xi_{i,t-1}y_{i,t-1}]}{var[\eta_i]}\hat{u}_{it}$$

Hence,

$$\underline{\xi}_{it} = \underline{\delta}_i + \underline{\omega}_{it} \tag{14}$$

$$\underline{\omega}_{it} = \underline{\psi}(L)\underline{\xi}_{it} \tag{15}$$

$$\underline{\omega}_{it} = \underline{\psi}(1)\underline{\xi}_{i,t-1} + \underline{\psi}(2)\underline{\xi}_{i,t-2} + \cdots + \underline{\psi}(T)\underline{\xi}_{i,0} \tag{16}$$

$$\underline{\xi}_{it} = \underline{\delta}_i + \underline{\psi}(1)\underline{\xi}_{i,t-1} + \underline{\psi}(2)\underline{\xi}_{i,t-2} + \cdots + \underline{\psi}(T)\underline{\xi}_{i,0} \tag{17}$$

$$E[\hat{\psi}(1)|\hat{\eta}_i] = \psi(1) + \underline{\delta}_i + \underline{\psi}(1)\underline{\xi}_{i,t-1} + \underline{\psi}(2)\underline{\xi}_{i,t-2} + \cdots + \underline{\psi}(T)\underline{\xi}_{i,0} \tag{18}$$

By symmetry the moving averages can be generalized for the following estimators of $\psi(2), \cdots, \psi(T)$. Two underlines represent the second iteration.

$$E[\hat{\psi}(2)|\hat{\eta}_i] = \psi(2) + \underline{\underline{\delta}}_i + \underline{\psi}(2)\underline{\xi}_{i,t-2} + \underline{\psi}(3)\underline{\xi}_{i,t-3} + \cdots + \underline{\psi}(T)\underline{\xi}_{i,0} \tag{19}$$

$$\vdots$$

$$E[\hat{\psi}(T)|\hat{\eta}_i] = \psi(T) + \underset{T}{\underline{\underline{\delta}}_i} + \underline{\psi}(T)\underline{\xi}_{i,0} \tag{20}$$

where $\underline{\underline{\delta}}_i$ means a *2th* iteration and $\underline{\underline{\delta}}_{\substack{i\\T}}$ means a *Tth* iteration. Plugging equations (18); (19) and (20) into equation (12) yields,

$$\begin{aligned} y_{it} = \eta_i + \beta y_{i,t-1} + &\\ &\psi(1) + \underline{\delta_i} y_{i,t-1} + \underline{\psi}(1)\underline{\xi}_{i,t-1} y_{i,t-1} + \underline{\psi}(2)\underline{\xi}_{i,t-2} y_{i,t-2} + \cdots + \underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0} + \\ &\psi(2) + \underline{\underline{\delta}}_i y_{i,t-2} + \underline{\psi}(2)\underline{\xi}_{i,t-2} y_{i,t-2} + \underline{\psi}(3)\underline{\xi}_{i,t-3} y_{i,t-3} + \cdots + \underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0} + \\ &\vdots \\ &\psi(T) + \underline{\underline{\delta}}_{\substack{i\\T}} y_{i,0} + \underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0} + u_{it} \end{aligned} \tag{21a}$$

Again, collecting the individual effects in a single term, *i.e.*, $\underline{\eta}_i = \eta_i + \delta_i y_{i,t-1} + \underline{\underline{\delta}}_i y_{i,t-2} + \cdots + \underline{\underline{\delta}}_{\substack{i\\T}} y_{i,0}$ leads to,

$$\begin{aligned} y_{it} = \underline{\eta}_i + \beta y_{i,t-1} + &\\ &\psi(1) + \underline{\psi}(1)\underline{\xi}_{i,t-1} y_{i,t-1} + \underline{\psi}(2)\underline{\xi}_{i,t-2} y_{i,t-2} + \cdots + \underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0} + \\ &\psi(2) + \underline{\psi}(2)\underline{\xi}_{i,t-2} y_{i,t-2} + \underline{\psi}(3)\underline{\xi}_{i,t-3} y_{i,t-3} + \cdots + \underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0} + \\ &\vdots \\ &\psi(T) + \underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0} + u_{it} \end{aligned} \tag{21b}$$

Now collecting similar terms together yields,

$$\begin{aligned} y_{it} = \underline{\eta}_i + \beta y_{i,t-1} + \psi(1) + \psi(2) + \cdots + \psi(T) + &\\ &\underline{\psi}(1)\underline{\xi}_{i,t-1} y_{i,t-1} + 2\underline{\psi}(2)\underline{\xi}_{i,t-2} y_{i,t-2} + \cdots + T\underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0} + u_{it} \end{aligned} \tag{22}$$

Consider that $\psi(1), \psi(2), \cdots, \psi(T)$ are individual fixed effects estimators. Consequently, they are the individual means at each lag value. The moving average terms can be collected with the specific individual-effects means, resulting in a single term, $\underline{\eta}_i$, that represents all individual effects in equation (21), *i.e.*, $\underline{\underline{\eta}}_i = \underline{\eta}_i + \psi(1) + \psi(2) + \cdots + \psi(T)$. Thus equation (22) can be rewritten as:

$$y_{it} = \underline{\underline{\eta}}_i + \beta y_{i,t-1} + \underline{\psi}(1)\underline{\xi}_{i,t-1} y_{i,t-1} + 2\underline{\psi}(2)\underline{\xi}_{i,t-2} y_{i,t-2} + \cdots + T\underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0} + u_{it} \tag{23}$$

Equation (23) represents the second iteration of the proposed novel recursive bias method to separate and quantify bias components. *Q.E.D.*

THEOREM 2: A consistent and efficient estimator can be computed for any panel data dimension size.

PROOF: Theorem 1 provides a recursive method for decomposing bias component estimators and recursively converge them with their efficient estimators. Thus, the following equality follows:

$$\frac{cov[\hat{\eta}_i, \xi_{i,t-1} y_{i,t-1}]}{var[\hat{\eta}_i]} \hat{u}_{it} = \\ \left[\left(\underline{\underline{\eta}}_i - \alpha_i \right) + \left(\underline{\psi}(1) \underline{\xi}_{i,t-1} y_{i,t-1} + 2\underline{\psi}(2) \underline{\xi}_{i,t-2} y_{i,t-2} + \cdots + T\underline{\psi}(T) \underline{\xi}_{i,0} y_{i,0} \right) \right] \quad (24)$$

where the left hand side is the estimator bias equation (3a), and the right hand side is the bias systematic component: $\underline{\underline{\eta}}_i - \alpha_i$, while the bias random component is: $\underline{\psi}(1)\underline{\xi}_{i,t-1} y_{i,t-1} + 2\underline{\psi}(2)\underline{\xi}_{i,t-2} y_{i,t-2} + \cdots + T\underline{\psi}(T)\underline{\xi}_{i,0} y_{i,0}$. Thus, after estimating equation (23), the following subtraction can be applied to equation (3a):

$$E[\hat{\beta}|\hat{\alpha}_i] = \beta + \frac{cov[\hat{\alpha}_i, y_{i,t-1}]}{var[\hat{\alpha}_i]} \hat{u}_{it} \\ - \left[\left(\underline{\underline{\eta}}_i - \alpha_i \right) + \left(\underline{\psi}(1) \underline{\xi}_{i,t-1} y_{i,t-1} + 2\underline{\psi}(2) \underline{\xi}_{i,t-2} y_{i,t-2} + \cdots + T\underline{\psi}(T) \underline{\xi}_{i,0} y_{i,0} \right) \right] \quad (25)$$

Hence, with this computation the estimator bias is reduced to zero, where β is an efficient estimator. It is evident that panel dimensions remain unchanged. This means that there is no need for either N or T, or both to be large. Therefore, the asymptotic bias properties are not needed for the recursive bias method.

$$E[\hat{\beta}|\hat{\alpha}_i] = \beta \quad (26)$$

Q.E.D.

5. ANALYTICAL FRAMEWORK, APPLICATION AND COMPARISON

The application uses the model of joint estimation of risk preference structure and technology using expo-power utility function (Saha et al. (1994)). This model has the flexibility to exhibit decreasing, constant or increasing absolute risk aversion and decreasing or increasing relative risk aversion depending on estimators values. The following equations (27-42) reproduce Saha et al. (1994) analytical approach equations.

5.1. *Analytical Framework*

In what follows is presented the relevant equations.

$$U(W) = \Theta - \exp(-\beta W^{\alpha}) \tag{27}$$

where $U(\cdot)$ denotes the utility function, exp denotes exponential, and W is wealth. Estimator restrictions of the expo-power utility function are $\Theta > 1$ and $\alpha\beta > 0$.

For the expo-power utility function, the Arrow-Pratt measures of absolute and relative risk aversion are respectively given by

$$A(W) = -U''(\cdot)/U'(\cdot) \tag{28}$$

$$R(W) = WA(W) \tag{29}$$

Under its estimator restrictions, the expo-power utility function exhibits DARA (Decreasing Absolute Risk Aversion) if $\alpha < 1$, CARA (Constant Absolute Risk Aversion) if $\alpha = 1$, and IARA (Increasing Absolute Risk Aversion) if $\alpha > 1$, DRRA (Decreasing Relative Risk Aversion) if $\beta < 0$, and IRRA if $\beta > 0$. Estimator Θ does not play any role in the characterization of the risk preference structure.

Technology is given by the following production function,

$$\tilde{Q} = h(x) + g(x, \varepsilon) \tag{30}$$

where x is an $n \times 1$ vector of inputs, $\tilde{Q}$ denotes random output with $h(x) : \Re^n \to \Re$, and $g(x, \varepsilon) : \Re^n \to \Re$. Random variable ε has support on $\Re$, and it captures production uncertainty.

Normalized random wealth, which is the sum of normalized random profit ($\tilde{\pi}$) and real initial wealth (I), is given by

$$\tilde{W} = \tilde{\pi} + I = h(x) + g(x, \varepsilon) - r^T x + I \tag{31}$$

where r denotes the $n \times 1$ vector or normalized input prices and T superscript denotes transpose. Using equation (31) the decision maker's problem of choosing optimal input levels to maximize expected utility can be stated as,

$$\begin{aligned} &\max_x H \equiv E[U(\tilde{W})] \\ &\max_x H \equiv E[U(h(x) + g(x,\varepsilon) - r^T x + I)] \end{aligned} \tag{32}$$

The n first-order conditions corresponding to n inputs are,

$$H_x \equiv E[U'(\cdot)\{h_x(\cdot) - r + g_x(\cdot)\}] = \underline{0} \tag{33}$$

where $\underline{0}$ is an $n \times 1$ vector of zeros, and the subscript x denotes derivatives. This set of equations can be written more compactly as

$$h_x(\cdot) - r + Z = \underline{0} \tag{34}$$

where $Z = E[U' g_x(x,\varepsilon)]/E[U']$. The second-order sufficient condition of (33) is negative definite,

$$H_{xx^T}(x^*) = h_{xx^T}(x^*) + Z_x(x^*) \tag{35}$$

The first-order equations of (34) are,

$$r = h_x(\cdot) + Z + e \tag{36}$$

where e denotes the vector of disturbances associated with 'errors' in optimization of the jointly estimations of the production, utility, and probability density functions.

The empirical model is based on equation (36). The proposed distribution of random variable in (30) is Weibull,

$$\varepsilon \sim \Omega(b,c) = \frac{c\varepsilon^{c-1}}{b^c} \exp\left\{-\left(\frac{\varepsilon}{b}\right)^c\right\}, \qquad +\infty \geq \varepsilon \geq 0 \tag{37}$$

where $\Omega(\cdot)$ denotes the probability density function of ε. The shape of the Weibull p.d.f. is determined by parameter c and its scale by b.

The production function, a combination of the Cobb-Douglas and exponential (CDE) forms is,

$$\tilde{Q} = A \prod_{i=1}^{n} x_i^{a_i} + \exp\left\{\sum_{i=1}^{n} m_i x_i + \varepsilon\right\} \tag{38}$$

where $A,\ a_i,\ m_i,\ i=1,\ldots,n$, are the $2n+1$ production technology estimators. In particular, if m_j, the *jth* coefficient in the stochastic part of the function, is negative, then the *jth* input is risk reducing in the sense that $\frac{\partial V(\hat{Q})}{\partial x_j} < 0$, where $V(\cdot)$ denotes variance. The Z term in the *jth* estimation of equation (36) can be written as,

$$Z_j = \frac{\displaystyle\int_o^{\infty} \alpha\beta W^{\alpha-1} \exp\{-\beta W^{\alpha}\} m_j \exp\left\{\sum_{i=1}^{n} m_i x_i + \varepsilon\right\} \frac{c}{b^c}\varepsilon^{c-1} \exp\left\{-\left(\frac{\varepsilon}{b}\right)^c\right\}\, d\varepsilon}{\displaystyle\int_o^{\infty} \alpha\beta W^{\alpha-1} \exp\{-\beta W^{\alpha}\} \frac{c}{b^c}\varepsilon^{c-1} \exp\left\{-\left(\frac{\varepsilon}{b}\right)^c\right\}\, d\varepsilon} \tag{39}$$

Under (38), the $h_x(\cdot)$ term for the jth estimation equation becomes

$$h_{x_j} = \frac{a_j\left\{A \prod_{i}^{n} x_i^{a_i}\right\}}{x_j} \tag{40}$$

Using (39) and (40), the *tth* observation on the *jth* equation in (36) can be reduced to,

$$r_{jt} = \frac{a_j\left\{A \prod_{i=1}^{n} x_{it}^{a_i}\right\}}{x_{jt}} + \frac{\displaystyle\int_o^{+\infty} W_t^{\alpha-1} \exp\left\{\sum_{i=1}^{n} m_i x_{it} + \varepsilon - \beta W_t^{\alpha} - \left(\frac{\varepsilon}{b}\right)^c\right\} \varepsilon^{c-1}\, d\varepsilon}{\displaystyle\int_o^{+\infty} W_t^{\alpha-1} \exp\left\{-\beta W_t^{\alpha} - \left(\frac{\varepsilon}{b}\right)^c\right\} \varepsilon^{c-1}\, d\varepsilon} m_j + e_{jt} \qquad j=1,\ldots,n,\ \ t=1,\ldots,T. \tag{41}$$

The system of n equations in (41) correspond to the number of observations. The log-likelihood function of the Weibull distribution is

$$\ln L(e/b,c) = T\ln(c) - cT\ln(b) + (c-1)\sum_{t}^{T} \ln\varepsilon_t - \left(\frac{\varepsilon_t}{b}\right)^c \tag{42}$$

5.2. *Application*

The empirical application uses data from a sample of Kansas wheat farmers. The results reject the null hypothesis of risk neutrality and suggest that Kansas farmers exhibit decreasing absolute risk aversion and increasing relative risk aversion.

Table I presents three sets of estimators, which are computed under MLE and a small-sample size of 60 observations. The first column is composed of synthetic estimators computed using the novel recursive bias method.[8] These synthetic estimators are taken from the first column of Table 3 in Carbajal-De-Nova (2021). The second column in Table I is the second set of estimators computed with real data and the expo-power utility method. Real data means data collected from farm surveys and administered registers. These real estimators are taken from the first column of Table 5 in Saha et al. (1994). The third column on Table I reports the third set of estimators computed with the traditional asymptotic bias method based on Monte Carlo simulations. They are drawn from the third column of Table I in Saha et al. (1997).[9]

[8]The construction of these synthetic estimators is described in Carbajal-De-Nova (2021).

[9]The Monte Carlo simulation design is available in this paper.

TABLE I

SYNTHETIC, REAL AND MONTE CARLO ESTIMATORS COMPARISON FOR THE EXPO-POWER AND STOCHASTIC PRODUCTION FUNCTION

		Estimates (Standard errors) [mean scuare errors]		
EP utility parameter	Explanation	Joint est.[a] Synthetic	Join est[a] Real	Join est.[b] Monte Carlo
α	$\alpha < 1 \rightarrow$ DARA	0.36 (0.02E-11)	0.36 (0.0294)	-0.10
β	$\beta > 0 \rightarrow$ IRRA	2.73 (0.01E-11)	2.73 (0.2201)	0.09
A	Parameters of the non-stochastic part of CDE[c]	1.60 (0.00)	1.60 (0.15)	1.20 [0.00]
a_1		0.25 (0.00)	0.25 (0.01)	0.29 [0.00]
a_2		0.75 (0.00)	0.75 (0.01)	0.60 [0.01]

[a] Expected utility maximization model (unrestricted).

[b] Monte Carlo experiments for group one using design matrix A, with 1,000 repetition sets of 60 observations. The initial values for these estimators are 0.86, 0.83, 0.87, 0.86, 0.86.

[c] Subindex 1 refers to capital, and 2 to materials. DARA stands for Decreasing Absolute Risk Aversion. IRRA stands for Increasing Relative Risk Aversion. CDE is the production function, a combination of the Cobb-Douglas and exponential forms, α and β reveals the risk preference structure.

5.3. *Estimators Comparison*

A comparison is made for the three sets of estimators reported in Table I. The real estimators by Saha et al. (1994) are the closest available estimates of the "true parameter value," since synthetic and Monte Carlo data do not have a real data generating process. The comparison of synthetic and Monte Carlo estimators against the "true parameter value" would reveal the efficiency of each method. Synthetic and real estimators are identical, since their difference is zero. Synthetic estimators standard errors are closer to zero “gaining efficiency” and thus have a bias reduction. Real and Monte Carlo estimator coefficients are quite dissimilar, where standard errors are not directly comparable with mean squared errors. This comparison demonstrated that the novel recursive bias method delivers unbiased estimators with identical coefficients to the "true parameter value."

Efforts to obtain unbiased estimators have been made by Abadie and Imbens (2011) in an empirical setting. They compute bias adjusted covariance matching (bacm) and bias adjusted propensity score matching (bapsm) estimators for experimental and nonexperimental data (Monte Carlo experiments with 10,000 repetitions). They use panel data analyzed originally by LaLonde (1986), with individual and time specific components, and a training dummy variable. Their non-experimental estimators do not reproduce the "true parameter value." Their bias reduction is small (around 0.01 percent) for one matching. For instance, their Monte Carlo bacm (1.43) and bapsm (1.64) standard errors are bigger than those belonging to experimental estimates (0.84) and (0.81), respectively.

The Arellano and Bond (1991) Monte Carlo simulation uses the "true parameter value" as seed. This value is 0.8 with a standard error of 0.048 (Table 4, column (c), first row). However, its Monte Carlo estimator is not identical nor more efficient than its empirical counterpart: 0.7827 with a standard error of 0.0582 (Table 1, third panel, column one, ninth and tenth rows).

Buccola and McCarl (1986) used Monte Carlo experiments with 1,200 replications and an execution time of twenty-five minutes to investigate small-sample properties of inputs on yield production functions. Their Table 1 reports the "true parameter value" and Monte Carlo estimators *i.e.*, 10 and 10.01 respectively, and their corresponding standard errors *i.e.*, 0.71 and 1.23. Thus, Monte Carlo estimators do not replicate the "true parameter value," neither report a smaller standard error.

6. CONCLUSIONS

Literature addressing a recursive bias method is scarce: Cornillon et al. (2014), Choi and Yang (2021), MacKinnon et al. (2023), Arellano and Bond (1991), Hsiao and Zhang (2015), Alvarez and Arellano (2003), Anderson and Hsiao (1981), Hsiao et al. (2002). These papers focus on the traditional asymptotic bias method to find efficient estimators, by increasing panel data dimensions N, T, or both. In contrast, the novel recursive bias finds efficient estimators closer to the "true parameter value" without enlarging N or T, or both. After analyzing Table I, it seems that the novel recursive fills a gap in the literature. As a result, the novel method could be a feasible option to provide efficient estimators for robust statistical inference and its decision making.

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