Minimum information copula under fixed Kendall's rank correlation

Issey Sukeda Tomonari Sei

Abstract

The minimum information copula (or the maximum entropy copula) is the most independent copula satisfying the given constraints. For these constraints, first-order expectation constraints on moments, such as Spearman's rank correlation, are mostly considered. On the other hand, such copulas under second-order constraints have not been studied well. We present a variant of minimum information copula that has a constraint on a popular second-order constraint known as Kendall's rank correlation, instead of first-order constraints. Due to this modification, the convexity of the problem becomes non-trivial and the form of density function of this variant is unknown. We analyze its property via one of the widely known discrete approximation of copulas, called checkerboard copulas. Checkerboard copulas can be considered identical to contingency tables. First, we introduce a transfer operation of probability mass on checkerboard copulas, which is technically equivalent to considering non-orthogonal basis of the total space of checkerboard copulas. Using this approach, we show several mathematical properties of the minimum information checkerboard copula under fixed Kendall's rank correlation. Firstly, this copula is characterized by a certain amount, which we name as "extended log odds ratio". It is also guaranteed that the density of this copula belongs to a function class known as "total positivity of order two (TP2)", one of the positive dependence properties that has been extensively studied for copulas. Furthermore, geometric interpretations of this problem setting will be investigated.

1 Minimum Information Copulas under fixed Kendall's rank correlation (MICK)

We consider an optimization problem where the information of a bivariate copula is minimized under the constraint fixing Kendall's rank correlation to a constant given in advance. We name its optimal solution MICK, which is the abbreviation of Minimum Information Copulas under fixed Kendall's rank correlation. For general moment conditions including Spearman's rank correlation, similar framework has been studied by [Bedford and Wilson(2014)], showing that the problem is convex. On the other hand, this problem for MICK is non-convex and its optimal solution is not obtained explicitly. Instead of solving the problem directly, we state several mathematical properties of MICK.

The continuous problem is written as follows:

$$(CP) \text{ minimize } \int_0^1 \int_0^1 p(x,y) \log p(x,y) \mathrm{d}x \mathrm{d}y,$$

$$\text{s.t. } \int_0^1 p(x,y)\mathrm{d}x = 1,\ \int_0^1 p(x,y)\mathrm{d}y = 1,$$

$$0 \leq p(x,y),$$

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\ \mathrm{sgn}(x-\tilde{x})\mathrm{sgn}(y-\tilde{y})p(x,y)p(\tilde{x},\tilde{y}) = \mu.$$

For simplicity, we we expect the discrete version asymptotically approach the continuous problem as the grid size grows.

$$
\begin{aligned}
(P) \text{ minimize } & \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{ij}\log \pi_{ij} \\
\text{s.t. } & \sum_{i=1}^{I} \pi_{ij} = \frac{1}{J}, \ \sum_{j=1}^{J} \pi_{ij} = \frac{1}{I}, \\
& 0 \le \pi_{ij} \le 1, \\
& 1 - \mathrm{tr}(\Xi\Pi\Xi\Pi^{\top}) = \mu,
\end{aligned}
$$

where $\mu \in [0,1]$ is a given constant. Note that we deal with the problem of checkerboard copula densities $p(x,y)$ through the problem of equivalent contingency tables (π_{ij}). The last constraint makes the problem non-convex.

2 Geometry around MICK

Piantadosi et al. [Piantadosi et al.(2012)Piantadosi, Howlett and Borwein] reformulate the representation of copulas using the fact that each doubly stochastic matrix is a convex combination of permutation matrices, thanks to Birkhoff-von Neumann theorem. Let $P_1 = [p_{1,ij}], \ldots, P_{n!} = [p_{n!,ij}] \in \mathbb{R}^{n\times n}$. There exists a convex combination

$$
\pi_{ij} = \sum_{k=1}^{n!} \alpha_k p_{k,ij}, \text{ such that } \sum_{k=1}^{n!} \alpha_k = 1, \alpha_k \ge 0.
$$

This representation, however, is redundant in that the combination is not determined uniquely. Instead, we attempt to represent checkerboard copulas according to a specific basis defined as follows.

Let $T_{ij} = (t_{ij}) \in \mathcal{M}_{n\times n}$, where

$$
t_{i'j'} = \begin{cases} 1 & \text{if } (i',j') = (i,j) \text{ or } (i',j') = (i+1,j+1), \\ -1 & \text{if } (i',j') = (i+1,j) \text{ or } (i',j') = (i,j+1), \\ 0 & \text{otherwise.} \end{cases}
$$

Then we observe that the space of checkerboard copula is expressed as a subspace of a $(n-1)^2$ dimensional vector space equipped with unorthogonal basis $\{T_{ij}\}$. For every checkerboard copula P,

$$
\exists 1\{p'_{ij}\} \in \mathbb{R}, P = U + \sum_{i=1}^{I-1}\sum_{j=1}^{J-1} p'_{ij} T_{ij}, \tag{1}
$$

where U denotes a uniform checkerboard copula. It is also convenient if we introduce a vector space corresponding to checkerboard copulas.

$$
p = u + \sum_{k=1}^{n^2} p'_k t_k = \frac{1}{n^2} 1_{n^2} + \sum_{k=1}^{n^2} p'_k t_k
$$

Here, p, u, t_k are n^2-dimensional vectors, aligning P, U, T_{ij} respectively. 1_{n^2} is an all-one n^2-dimensional vector Note that in order to assure P is a checkerboard copula, there are explicit constraints on p'_{ij} so that no entry in P becomes negative. From this equation, it is also possible to obtain (p'_{ij})s from (p_{ij})s.

$$
p' = (A \otimes A)^{\dagger}(p - \frac{1}{n^2} 1_{n^2}),
$$

where † denotes pseudo-inverse matrix, $\otimes$ denotes Kronecker product, and $n \times (n-1)$ matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \dots \\ 0 & 0 & -1 & \dots \end{pmatrix}.$$

For example, $p'_{ij} = 0$ $(\forall i, j)$ for a uniform copula and

$$\begin{cases} p'_{11} = p'_{44} \risingdotseq 0.0791 \\ p'_{22} = p'_{33} \risingdotseq 0.1445 \\ p'_{12} = p'_{21} = p'_{34} = p'_{43} \risingdotseq 0.0949 \\ p'_{13} = p'_{31} = p'_{24} = p'_{42} \risingdotseq 0.0734 \\ p'_{14} = p'_{41} \risingdotseq 0.0386 \\ p'_{23} = p'_{32} \risingdotseq 0.1304 \end{cases}$$

for MICK with the grid size of 5×5.

Now, we provide an intuitive interpretation of $\{T_{ij}\}$ from a different perspective. From the definition of T_{ij}, increasing the coordinate p'_{ij} in equation (3) means to first choose any 2×2 region on a checkerboard copula P and then transfer probability mass from two diagonal regions to the other adjacent regions. In other words, its diagonal regions increase by Δ while its anti-diagonal regions increase by $-\Delta$, keeping its row sum and column sums the same. This movement guarantees that P is still a checkerboard copula after the transfer. When you start from an uniform copula and try to apply this transfer as many times as possible on a copula, you will eventually arrive at the comonotone copula.

With this new basis $\{T_{ij}\}$, the space of checkerboard copulas can be visualized for better understanding of their properties. The space of discrete $I \times J$ bivariate copulas is known to corresponds to a polytope called generalized Birkhoff polytope. When $I = J = n$, it corresponds to a Birkhoff polytope, noted as $\mathcal{B}_n$. $\mathcal{B}_n$ has $n!$ vertices, each of which corresponds to a permutation matrix. Therefore, the optimization problem (P) is a problem where we find a minimum information discrete distribution on intersection of $\mathcal{B}_n$ and K, where K denotes the curve surface with a constant Kendall's rank correlation.

Example 1 $(I = 3, J = 2)$**.** Expression of a copula P in new coordinates is

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \\ p_{31} & p_{32} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} + p'_{11} \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} + p'_{21} \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The constraint on Kendall's tau becomes $1 - \mathrm{tr}(\Xi P \Xi P^\top) = \mu \Leftrightarrow \frac{4}{3}p'_{11} + \frac{4}{3}p'_{21} = \mu$, which is depicted as a green line in Figure 1.

3 Main results

In this section, we observe that there exists a invariant value on every 2×2 submatrices of the matrix associated with MICK by considering the stationary conditions. First, we derive two values, the variation of Kendall's tau and the variation of the objective function of (P). Proofs are just tedious calculations, thus omitted in this paper.

Lemma 1 (Variation of Kendall's tau)**.** *Let* $\Pi = (\pi_{i,j})$ *be a checkerboard copula. Consider a small change* $\epsilon T_{ij} (\epsilon \ll 1)$ *on* Π*. The variation of Kendall's tau is*

$$2\epsilon(\pi_{i,j} + \pi_{i+1,j+1} + \pi_{i+1,j} + \pi_{i,j+1})$$

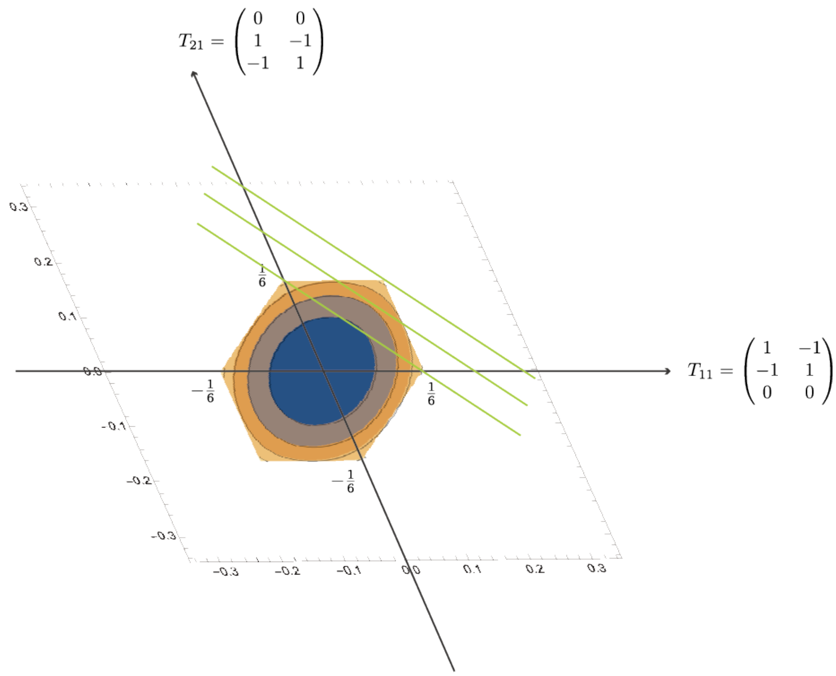

Fig. 1: Visualization of 3×2 checkerboard copulas space

This value $\pi_{i,j} + \pi_{i+1,j+1} + \pi_{i+1,j} + \pi_{i,j+1}$ is positive, meaning that Kendall's tau always increases when T_{ij} is added to a copula.

Lemma 2 (Variation of the objective function)**.** *Let $\Pi = (\pi_{i,j})$ be a checkerboard copula. Consider a small change $\epsilon T_{ij} (\epsilon \ll 1)$ on Π. The variation of copula information is*

$$\epsilon \log \frac{\pi_{i,j}\pi_{i+1,j+1}}{\pi_{i+1,j}\pi_{i,j+1}}.$$

Two variations from Lemma 3 and Lemma 4 leads to the following main statement.

Theorem 1. *The following value is constant for every 2×2 submatrices $\begin{pmatrix} \pi_{i,j} & \pi_{i,j+1} \\ \pi_{i+1,j} & \pi_{i+1,j+1} \end{pmatrix}$ on MICK.*

$$\frac{1}{\pi_{i,j} + \pi_{i+1,j} + \pi_{i+1,j} + \pi_{i+1,j+1}} \log \frac{\pi_{i,j}\pi_{i+1,j+1}}{\pi_{i+1,j}\pi_{i,j+1}}$$

We name this common value "extended log odds ratio".

Proof of Theorem 5: The stationary condition for the optimization problem (P) is

$$\forall \epsilon (\ll 1), i, j, i', j', \epsilon \log \frac{\pi_{i,j}\pi_{i+1,j+1}}{\pi_{i+1,j}\pi_{i,j+1}} - \delta \log \frac{\pi_{i',j'}\pi_{i'+1,j'+1}}{\pi_{i'+1,j'}\pi_{i',j'+1}} = 0$$

Therefore, we obtain

$$\frac{\pi_{i',j'} + \pi_{i'+1,j'} + \pi_{i'+1,j'} + \pi_{i'+1,j'+1}}{\pi_{i,j} + \pi_{i+1,j} + \pi_{i+1,j} + \pi_{i+1,j+1}} \log \frac{\pi_{i,j}\pi_{i+1,j+1}}{\pi_{i+1,j}\pi_{i,j+1}} - \log \frac{\pi_{i',j'}\pi_{i'+1,j'+1}}{\pi_{i'+1,j'}\pi_{i',j'+1}} = 0$$

Hence,

$$\frac{\log \frac{\pi_{i,j}\pi_{i+1,j+1}}{\pi_{i+1,j}\pi_{i,j+1}}}{\pi_{i,j} + \pi_{i+1,j} + \pi_{i+1,j} + \pi_{i+1,j+1}} = \frac{\log \frac{\pi_{i',j'}\pi_{i'+1,j'+1}}{\pi_{i'+1,j'}\pi_{i',j'+1}}}{\pi_{i',j'} + \pi_{i'+1,j'} + \pi_{i'+1,j'} + \pi_{i'+1,j'+1}}.$$

Note that both sides of the equation has the same form for (i, j) and (i', j'). $\square$

For comparison, the similar derivation can be applied to minimum information copula under fixed Spearman's rho, studied by Piantadosi et al. [Piantadosi et al.(2012)Piantadosi, Howlett and Borwein]. In this case, the extended log odds ratio coincides with the normal log odds ratio since the variation of Spearman's rho by the mass transportation T_{ij} becomes a constant. Furthermore, we can explicitly calculate its log odds ratio and confirm this fact because the form of the optimal solution is known in previous studies:

$$\pi_{ij} = A_i B_j \exp\left(12\theta(i - \frac{1}{2})(j - \frac{1}{2})\right).$$

It is easy to check

$$\log \frac{\pi_{ij}\pi_{i+1,j+1}}{\pi_{i+1,j}\pi_{i,j+1}} = \theta \ (constant).$$

4 Total Positivity

Total positivity(TP) is one of the notions of positive dependence, extensively studied for copulas and other general functions.

Definition 1 (TP2). We say a function $f : \mathbb{R}^2 \to \mathbb{R}$ belongs to TP2 when $f(x,y) \geq 0$ and $\begin{vmatrix} f(x,y) & f(x,y') \\ f(x'y) & f(x',y') \end{vmatrix} \geq 0 \ (x < x', y < y')$. For matrices, it is called TP2 if its minors (determinant of principle submatrices) of order 2 are non-negative.

The following lemma is direct from the definition of TP2.

Lemma 3. *A matrix is TP2 if and only if its 2×2 submatrices are all TP2.*

A copula C is said to be TP2 if the copula itself belongs to TP2:

$$C(x,y)C(x',y') - C(x,y')C(x',y) \geq 0 \ (x < x', y < y'),$$

and d-TP2 (density TP2) if its density c satisfies

$$c(x,y)c(x',y') - c(x,y')c(x',y) \geq 0 \ (x < x', y < y').$$

Theorem 2. *MICK belongs to d-TP2.*

Proof of Theorem 2: Let a copula $P^* = (p^*_{ij})$ be a MICK. All entries are positive. From Theorem 8, P^* is also an optimal solution of (RP). Assume P^* is not d-TP2. Then, there exists a 2×2 submatrix of P^* such that its determinant is negative:

$$\exists\ i,j \in \{1,\ldots,n-1\},\ p^*_{ij}p^*_{i+1j+1} - p^*_{i+1j}p^*_{ij+1} < 0 \Leftrightarrow \log \frac{p^*_{ij}p^*_{i+1j+1}}{p^*_{i+1j}p^*_{ij+1}} < 0.$$

From Lemma 1, 2 and 4, this indicates $P^* + T_{ij}$ achieves smaller value and larger Kendall's tau than P^*. This contradicts with the optimality of P^*. □

References

[Bedford and Wilson(2014)] T. Bedford, K. J. Wilson, On the construction of minimum information bivariate copula families, Annals of the Institute of Statistical Mathematics 66 (2014) 703–723.

[Piantadosi et al.(2012)Piantadosi, Howlett and Borwein] J. Piantadosi, P. Howlett, J. Borwein, Copulas with maximum entropy, Optimization Letters 6 (2012) 99–125.